[27] Chaoran Yang and John Mellor-Crummey. A wait-free queue as fast as fetch-and-add. *SIGPLAN Not.*, 51(8):16:1–16:13, February 2016. ISSN 0362-1340. doi: 10.1145/3016078.2851168. URL http://doi.acm.org/10.1145/3016078.2851168.

A A TIGHTER VERSION OF LEMMA 3.3.

We now prove a slightly tighter version for the case where only elements from the top-k may be chosen by the scheduler Q. Note that the condition in the following Lemma (that u and v are simultaneously in the top-k) is a pre-requisite for u to experience an inversion on or above v, and thus the Lemma is slightly stronger than necessary.

Lemma A.1. *Consider running Algorithm 2 (or Algorithm 4) using a k-relaxed queue Q on input graph $G = (V, E)$. For a fixed edge $e = (u, v)$, the probability that both u and v are simultaneously in the top-k of Q during any execution on a random permutation π is bounded by $O(k^2/n)$.*

Proof. We will write $e \in$ top-k as shorthand for the event that u and v are simultaneously in the top-k of Q at some time. Note that no matter what dependencies exist in the top-k of Q, the entire top-k is flushed after the rank 1 element gets deleted k times. The number of iterations it takes to delete the rank 1 element k times after u enters the top-k (thereby flushing u with certainty) is a negative binomially distributed random variable X_u with mean k^2 and success probability $1/k$ (due to the fairness of Q), and similarly for X_v. Since S_e entails that either $\ell(u) < \ell(v) < \ell(u) + X_u$ or $\ell(v) < \ell(u) < \ell(v) + X_v$, we note that the two cases are symmetric and compute:

$$
\begin{aligned}
\Pr[e \in \text{top-k}] &\le \Pr[\ell(u) < \ell(v) < \ell(u) + X_u] \\
&\quad + \Pr[\ell(v) < \ell(u) < \ell(v) + X_v] \\
&= 2\Pr[\ell(u) < \ell(v) < \ell(u) + X_u] \\
&= 2\sum_r \Pr[X_u = r]\Pr[\ell(u) < \ell(v) < \ell(u) + r] \\
&= 2\sum_r \Pr[X_u = r]\,\frac{r}{n} \\
&\le \frac{2ck^2}{n}\Pr\left[X_u \le ck^2\right] + 2\sum_{r > ck^2}\Pr[X_u = r]\,\frac{r}{n} \\
&\le O\left(\frac{k^2}{n}\right) + 2\sum_{r'}\Pr\left[X_u > r'k^2\right]\frac{(r'+1)k^2}{n} \\
&\le O\left(\frac{k^2}{n}\right)\left(1 + \sum_{r'} e^{O(-r')}(r'+1)\right) \qquad (*) \\
&= O\left(\frac{k^2}{n}\right),
\end{aligned}
$$

where (*) uses a standard tail bound on the Negative Binomial Distribution[2]. □

[2] See [16] for a derivation.