# Pragmatic Primitives for Non-blocking Data Structures PODC 2013

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Trevor Brown Pragmatic Primitives for Non-blocking Data Structures

Data structures that can be accessed concurrently by many processes are

- important
- hard to design
- hard to prove correct

We focus on linearizable, non-blocking data structures.

#### Transactional memory

Enclose each data structure operation in an atomic transaction.

Pros:

- simple to design
- simple to prove correct

Cons:

- less efficient than hand-crafted data structures
- coarse-grained transactions limit concurrency

Right solution for "casual" data structure designers.

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Handcrafted non-blocking implementations from hardware primitives.

Pros:

- allows good efficiency
- allows high degree of concurrency

Cons:

hard to get implementation (provably) right

Right solution for designing libraries of data structures.

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## Why is it hard to use hardware primitives?

Key difficulty of implementing data structures from hardware primitives:

- Data structure operations access several words atomically
- Hardware primitives operate only on single words

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Problems arise if we concurrently INSERT(B,3) and DELETE(B,2).



#### Each operation prepares to do its CAS.

- INSERT occurs
- DELETE occurs, three copies of *B* are lost.

DELETE should succeed only if node *B* is unchanged.

Need multi-word primitives.

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Build "medium-level" primitives that can access multiple words.

- higher-level than CAS or LL/SC
- lower-level than full transactional memory

Advantages:

- General enough to be used in many data structures
- Specialized enough to create quite efficient implementations
- Modular proof of correctness: large parts can be reused

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Our primitives work on data records.

Each data record has

- some mutable fields (one word each)
- some immutable fields

Use a data record for some natural "unit" of a data structure

- node in a tree
- entry in a table

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Our primitives extend load-link (LL) and store-conditional (SC).

#### LL/SC object

- stores a single word
- LL reads value stored
- SC(v) (store-conditional) writes v only if value has not changed since last LL by process performing SC.

LLX(r) returns a snapshot of the mutable fields of r

SCX(V, R, field, new) by process p

- writes value *new* into *field*, which is a mutable field of a data record in V
- finalizes all data records in  $R \subseteq V$
- only if no record in V has changed since p's LLX on it

After a data record is finalized, no further changes allowed.

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### Example: removing all copies of a key in multiset

DELETE(B, 2) using LLX and SCX. Use one data record for each node.



- $\rightarrow \langle \textbf{B.count} = \textbf{2}, \textbf{B.next} = \textbf{D} \rangle$
- - changes A.next to D
  - finalizes B
  - succeeds only if no changes since LLXs on A and B

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## Example: removing all copies of a key in multiset

DELETE(B, 2) using LLX and SCX. Use one data record for each node.



- LLX(A)  $\rightarrow \langle A.count = 3, A.next = B \rangle$
- 2 LLX(*B*)
  - $\rightarrow \langle \textit{B.count} = 2, \textit{B.next} = \textit{D} \rangle$
- **3** SCX( $\langle A, B \rangle, \langle B \rangle, A.next, D$ )
  - changes A.next to D
  - finalizes B
  - succeeds only if no changes since LLXs on A and B

- Large LL/SC objects (Anderson Moir 1995, ...)
   ⇒ unable to access multiple objects atomically
- Multi-word CAS (Israeli Rappoport 1994, ...)
   ⇒ more general, less efficient
- Multi-word RMW (Afek Merritt Taubenfeld Touitou 1997, ...)
   ⇒ even more general, less efficient
- *k*-compare-single-swap (Luchangco Moir Shavit 2009)
   ⇒ lacks ability to finalize
  - $\Rightarrow$  less efficient for some applications

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LLX(r) can return one of the following results.

- a snapshot of mutable fields of r
- FINALIZED (iff *r* has been finalized by an SCX)
- FAIL (in our implementation this happens only if a concurrent SCX accesses *r*)

We also allow reads of individual mutable fields.

Before calling SCX(V, R, field, new), process p must get a snapshot from an LLX(r) on each record r in V.

For each r in V, the last LLX(r) by p is linked to the SCX.

If any *r* in *V* was changed since the linked LLX(*r*)  $\Rightarrow$  SCX returns FAIL.

Non-failed SCX sets *field*  $\leftarrow$  *new* and *finalizes* records in *R*.

Spurious failures of SCX are allowed.

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Individual LLXs and SCXs are wait-free, but may fail.

- If LLXs and SCXs are performed infinitely often, they succeed infinitely often.
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Also, if no overlap between V-sets of SCX's, all will succeed.

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#### Lock-free Locks

- "Locks" on data records acquired by SCX operations
- If a record you need is locked by another SCX, you can help that SCX and release lock
- Finalized records remain permanently locked

Based on cooperative technique of Turek et al. [1992] and Barnes [1993]

Each SCX creates an SCX record.

An SCX record contains all information needed to help SCX.



Add two fields to each data record *r*:

- info: pointer to SCX record of last SCX that locked r
- marked: boolean used to finalize r



Expected value for CAS comes from LLX.  $\Rightarrow$  CAS succeeds only if info field unchanged since LLX

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```
SCX(V, R, field, new)

create SCX record s

for each r in s.V

[lock r]

CAS s.new into s.field

s.state \leftarrow committed

end SCX
```

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```
\frac{SCX(V, R, field, new)}{create SCX record s} HELP(s)
for each r in s.V
[lock r]
CAS s.new into s.field
s.state \leftarrow committed
end SCX HELP
```

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# Help algorithm

```
HELP(s)
   for each r in s V
      try to CAS s into r.info
      if r.info \neq s then
         if s.locksSucceeded then
            return TRUE % Someone else finished the operation
         else
            s.state \leftarrow aborted
            return FALSE
   s.locksSucceeded ← TRUF
   r.marked \leftarrow TRUE for each data record in R
   CAS s new into s field
   s.state \leftarrow committed
   return TRUE
end HELP
```

- Locking correctly protects all mutable fields of a record
- All helpers of an SCX agree on outcome (failed/succeeded)
- No ABA problems on fields accessed by CAS
- Progress properties

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- p locks A, B q locks B, A
  - 2 Real locks: deadlock!
  - Lock-free locks: abort & retry, but repeat forever ⇒ livelock!

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- Peal locks: deadlock!

Lock-free locks: abort & retry, but repeat forever ⇒ livelock!

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Need SCXs to "lock" in consistent order  $(\Rightarrow \text{ one will eventually succeed})$ 



- p locks A, B q locks B, A
- 2 Real locks: deadlock!
- Sock-free locks: abort & retry, but repeat forever ⇒ livelock!

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Need SCXs to "lock" in consistent order ( $\Rightarrow$  one will eventually succeed)

#### Constraint

After SCX's stop succeeding, eventually all new SCX's must have consistent order on V-sets.

Easy to satisfy, because you can ignore concurrency.

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With no contention: SCX performs

- k + 1 CAS steps if it depends on k LLXs
- *f* + 2 writes if it finalizes *f* data records

LLX only performs reads.

With contention, LLXs and SCXs may have to help and/or retry.

Future work: Amortized complexity bounds with contention.

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# Summary

Contributions:

- Semantics of LLX and SCX (could be implemented, e.g., with HTM)
- Vastly simplifies proofs of correctness for non-blocking data structure implementations

Further work:

- VLX (generalizes validate instruction)
- Non-blocking balanced BSTs (and template for building other trees)
- Experimental results

#### **Extra slides**

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## When k-compare-single-swap (kCSS) is inefficient

Example: tree where each node has 32 child pointers (or keys).



Requires a 33-compare-single-swap operation

With no contention:

- kCSS: 2 CASs, 2 writes, 66 non-cached reads
- SCX+LLXs: 2 CASs, 1 write, ≤ 13 non-cached reads

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